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Response of Elastomers to Any Forcing Function

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INTRODUCTION

A recent article[1] points out that at least some fifty papers have been published on the subject of the dynamic properties of high polymers. These papers are usually written with the objective of assessing the quality of materials used in tires, vibration mountings, shock absorber stops, rubber stops, etc. Throughout these papers, it turns out that damping is expressed in at least 14 different units.[2] Our view is that a large part of the confusion arises from the attempt to analyze rubber-like behavior in terms of the second order system:

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega t \qquad (1)$$

where x is a measure of the strain and $\dot{x}$ and $\ddot{x}$ are its derivatives, t is time, P_0 is the amplitude of the applied force, and ω is the radian frequency of force application.

If c and k were constant over the range of x and ω, there would be little difficulty. However, such is not the case. Figures 1 and 2 show the variation of the spring rate k and damping coefficient c with frequency for the case of compression forcing in rubber. These data are those of Cooper[2] of Dunlop. The attempt to apply a simple model such as that expressed by eq. (1) to a system as complex as that characterized by Figures 1 and 2 is bound to lead to confusion. In addition, even if the current analysis could be made by means of correct sinusoidal analysis, how could one predict the response of the elastomer to the specific forcing to which the rubber article would be subjected in the tire or shock mounting? These forcing functions are quite complex and contain a large number of frequency components. For example, remembering Figure 1, how can one specify the damping characteristics of a rubber article when it is acted upon by a forcing function containing a wide variety of frequency components?

The method described in this paper, we feel, represents a different approach to this problem. In its most sophisticated form, no assumptions are made concerning models and the response of the rubber to almost any forcing function can be calculated.

ANALYSIS

Before going into the experimental details of the technique, I would like to review briefly some of the mathematical justification for this method.

A widely used theorem in the field of electrical analysis is given by

$$f_3(t) = d/dt\,[\textstyle\int_0^t f_1(t - \lambda)f_2(\lambda)d\lambda] \qquad (2)$$

where λ is an integration variable, $f_1(t)$ is the response of a system to a unit step function and $f_3(t)$ is the response of the system to the forcing function

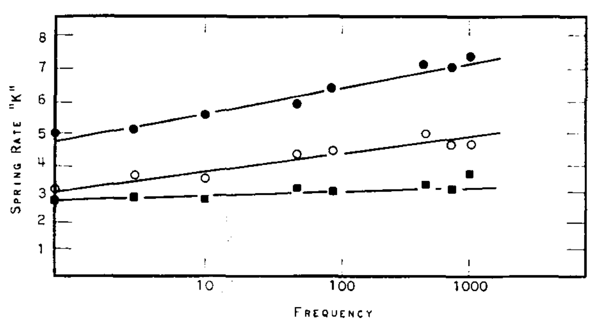

Fig. 1. Spring rate against frequency.

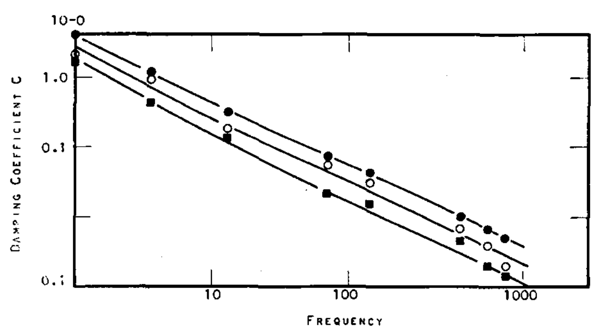

Fig. 2. Damping against frequency.

$f_2(t)$. Another way of stating eq. (2) is that if we know the response of a system to a step function then we can predict its response to any other function. The application of this theorem to the study of elastomers is quite easy to see. If we rewrite eq. (2) in the usual nomenclature of elastomeric systems and simplify, we get

$$e(t) = f_1(t)S(0) + \int_0^t f_1(t-\lambda)\ [dS(\lambda)/dt]d\lambda \quad (3)$$

where now $f_1(t)$ is the strain obtained under a step force and $e(t)$ is the strain response under a force $S(t)$. In other words, if we know the deformation of a sample of rubber to a step force, then we can predict its deformation to any force.

Mathematically, there is a limitation to this technique, and that is that the dynamic system describing the elastomeric material be linear. (We will see later how this assumption can be removed.) However, we need not make any other assumption about the system. All we need to know is the response of the elastomer to a step force and then we can predict its response to any force.

EXPERIMENTAL

As the analysis above points out, in order to make practical use of the theory, we must be able to apply a step force experimentally to the rubber. Of course, it is impossible to apply a true mathematical step, for this would mean applying a force in no time at all. However, we must apply a force in a time which must be faster than the response of the rubber and much faster than the highest frequency component we wish to study. That is to say, on the time scale we wish to analyze the forcing function must look like a step. The experimental setup we finally arrived at is shown in Figures 3 and 4. The wire cutter cuts the wire.

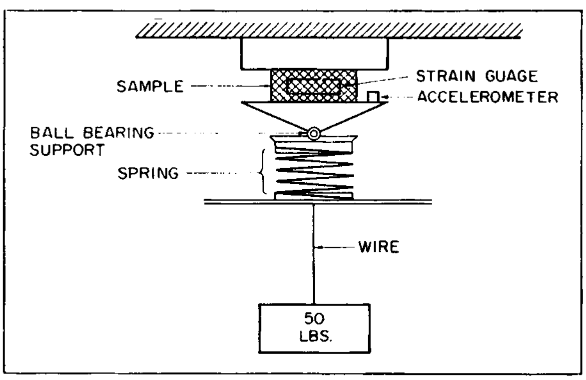

Fig. 3. Schematic of the mechanical components of the step function machine.

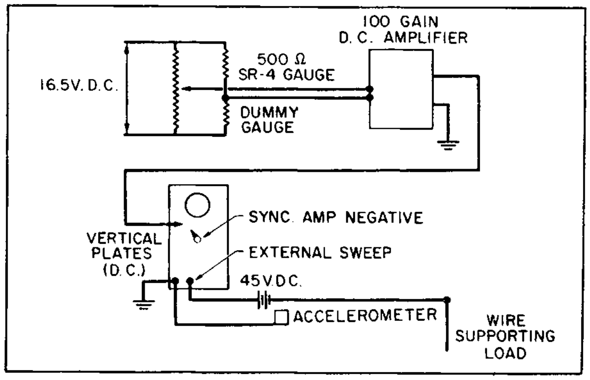

Fig. 4. Strain gauge and sweep circuit for step function machine.

This action releases the weight and the spring transfers the load to the rubber sample. Thus, the rubber sample receives a step force in compression. Strain gages on the sample measure the deformation of the rubber. Hence, we have the response of the rubber to a step force. With this information we are prepared to apply the analysis of the previous section. Before doing this, however, I would like to discuss some of the experimental details. In order for the spring to exert most of the load on the rubber sample, the spring must be much softer than the rubber. If it is not, the deflection of the rubber will relieve the load exerted on it by the spring. The bonded strain gages on the rubber which detect the rubber response must be calibrated in the step machine. This is done by applying various loads to the rubber, measuring the deflection of the rubber by means of a dial gage, and noting the output of the strain gages. A typical calibration is shown in Figure 5. As shown in the diagram, an accelerometer triggers the sweep of the scope so that the response signal can be seen. Figure 6 shows how successful we were in applying a true step. (The time between arrows represents 50 μsec.) This signal was ob-

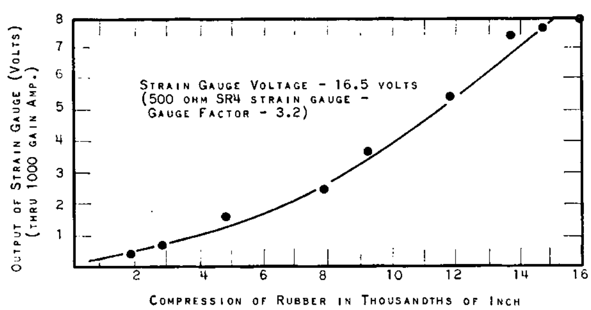

Fig. 5. Calibration of rubber–strain gauge bond.

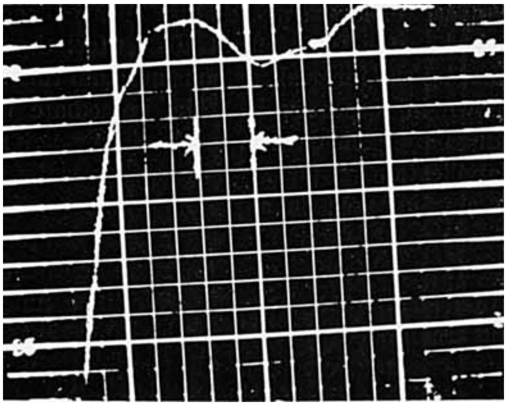

Fig. 6. Step force of 50 lb.

tained from the accelerometer on the platform supporting the sample. For most of the results shown here, the signal was triggered by other means. Unfortunately, this triggering was not quick enough to catch the response curve at about zero time. However, the accelerometer technique described above does so adequately. One further point should be made concerning the experimental setup. At rates such as those we are using here with a mechanical setup, it is virtually impossible to eliminate inertial effects. Our best calculation is that we have about 150 g. of mass involved in the dynamic system. This is not as serious a limitation as it seems at first glance. What this means is that rather than the response of the elastomer alone, we obtain the response of the elastomer–mass system. However, since the mass is constant for all time, the inertial effect is not a very serious detriment in evaluating polymers. As will be shown later, it is possible to eliminate inertial effects by special techniques.

DISCUSSION

Figure 7 shows a typical response of an elastomer to the step function as obtained by our machine. The time interval indicated by one division is approximately 2 msec. The inertia effect is clearly seen by the undulations at the top of the curve. Notice also that in these photographs we have not caught the initial rise time of the response curves. This is an important portion of the curve, since it corresponds to high frequency behavior. As mentioned earlier, we have improved our "triggering" technique and are now in a position to study the complete response curve. Once these curves are obtained, they must be inserted into the integral eq. (3) together with the forcing function one is

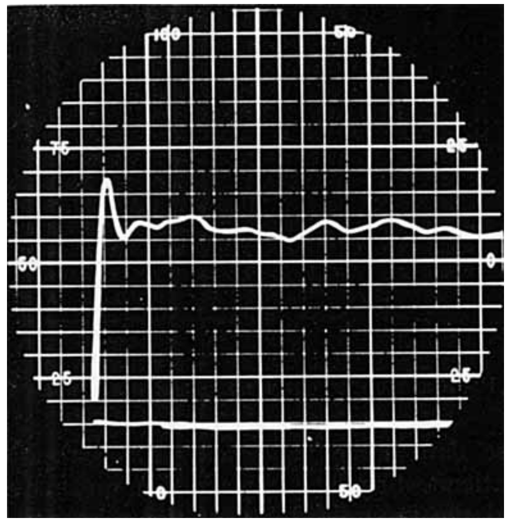

Fig. 7. Typical response to a step force.

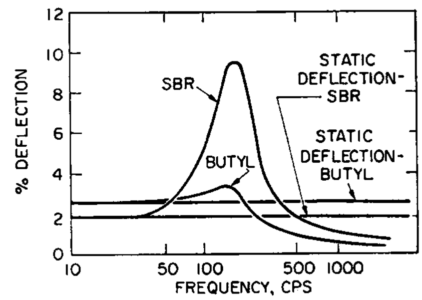

Fig. 8. Frequency response of butyl and SBR vulcanizates as calculated from step function. Driving force was 50 lb. sin 2 π ft.

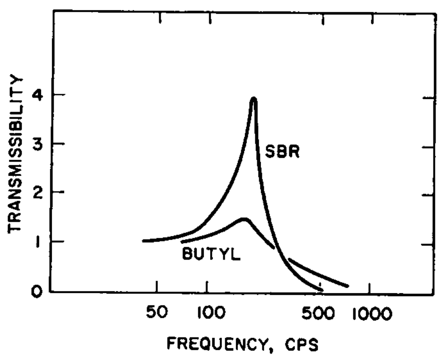

Fig. 9. Transmissibility of butyl and SBR as a function of frequency.

interested in. To check out the method, we first studied sinusoidal forcing. The method of integrating the equation is primarily that used by Schechter et al.[3] We feel we have improved on this technique, and it is discussed in detail in Appendix I. Figure 8 shows the calculated response of unfilled butyl and SBR rubbers over a wide frequency range.

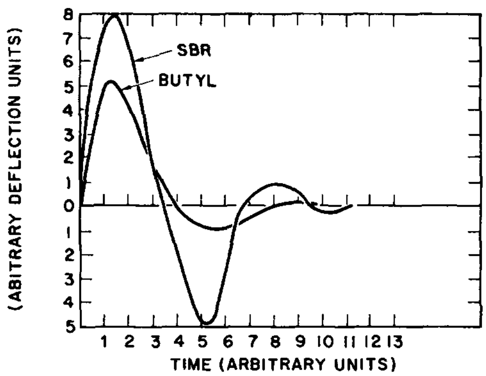

Fig. 10. Response of butyl and SBR to a pulse force.

These data agree very well with conventional vibration testing data. In addition, we have computed the transmissability of the polymers over this frequency range. Finally, we have been able to calculate how these polymers would respond to a pulselike blow as would occur in shock loading. This is shown in Figures 9 and 10. A technique for evaluating the integral for nonsinusoidal functions is shown in Appendix II. The immediate value obtained by such data is in classifying polymers for application purposes. If one knows the type of forcing imposed on a polymer in a particular application, the response of various compounds under identical forcing conditions can be calculated by this technique. Another advantage inherent in this technique is that the test is so fast that there is no temperature rise in the sample. In high hysteresis stocks, particularly, temperature rise during dynamic testing is very difficult to control.

USE OF ANALOGS

Recently, we have been using an analog computer in conjunction with the step function work.

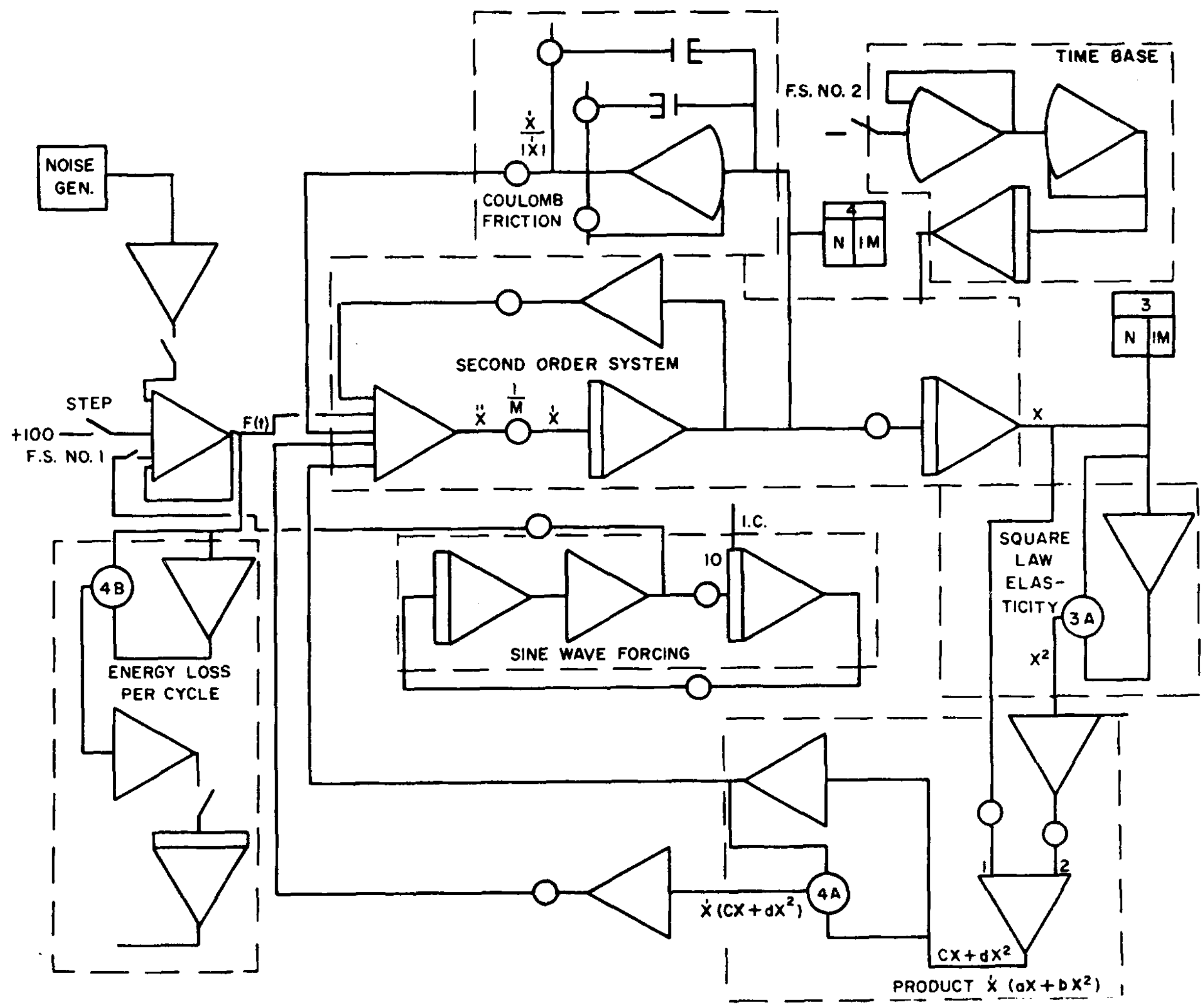

Fig. 11. General polymer analog electrical circuit. The function of each of the elements is shown within the dashed lines.

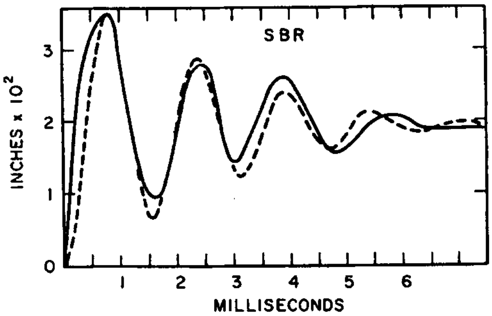

Fig. 12. Comparison of step function data with analog response for SBR: (- -) analog; (——) data.

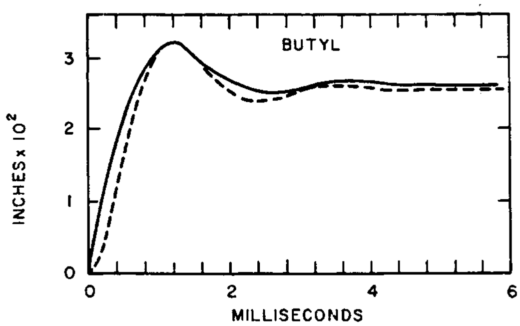

Fig. 13. Comparison of step function data with analog response for butyl: (- -) analog; (——) data).

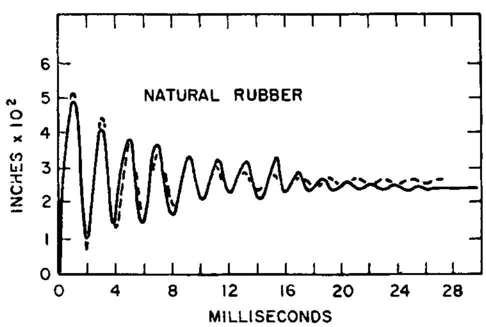

Fig. 14. Comparison of step function data with analog response for natural rubber: (- -) analog; (——) data.

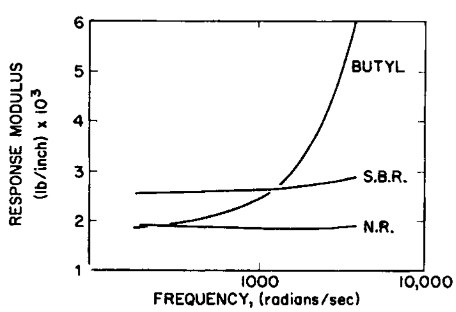

Fig. 15. Effective modulus (the ratio of maximum force to maximum displacement) for butyl, SBR, and natural rubber. These tests were run over a frequency range of 100–10,000 radians/sec.

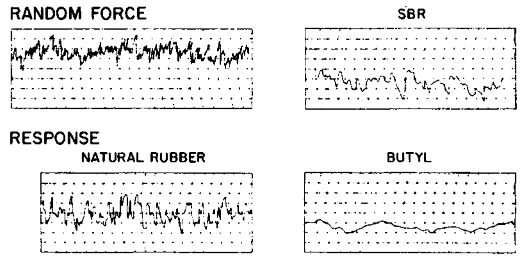

Fig. 16. Response of butyl, SBR, and natural rubber to random force. The frequency content of the random force is 0–2000 cycles/sec. The steadier the response, the less rubberlike the behavior.

This increases the power of the technique. To a large extent, the linearity restriction is removed. The procedure is simply to set up a mechanical-electrical analog of the rubber such that the response of the analog to a step function is the same as that of the rubber. The components of the analog should then bear a direct relationship with the dynamic variables of the elastomers. In addition, the inertial terms can be eliminated so that the rubber can be studied per se. Figure 11 shows the analog circuit. Figures 12, 13, and 14 show the reponse of SBR, butyl, and natural rubber and the respective responses of their analogs. It can be seen that the analogs are faithful reproductions of the elastomers except at the initial portion of the response. The implication of this will be discussed later. Based on these analogs, the moduli of the three rubbers were measured and are shown in Figure 15. In addition, the response of the elastomers to random forces (frequency range 0–6000 cycles) is calculated and shown in Figure 16. This type of information is very useful from an application point of view. Random forces such as shown in Figure 16 probably occur in a tire tread going over a rough road or a vibration isolater under a piece of factory equipment.

A study of the analogs themselves indicates that the following dynamic equations govern the behavior of the elastomers:

SBR rubber:

$$m(d^2x/dt^2) + [2/(dx/dt)](dx/dt) + 0.35(dx/dt) + 2590X = 0$$

Butyl rubber:

$$m(d^2x/dt^2) + 1.4(dx/dt) + 1900X = 0$$

Natural rubber:

$$m(d^2x/dt^2) + 0.25(dx/dt) + [1 - 0.0021(dx/dt)] 1950X = 0$$

All the polymers appear to be represented generally as a simple spring-dashpot system. However, there are some deviations in both SBR and natural rubber. In SBR, the energy loss seems to be of two types. One is the usually rate-dependent viscosity loss, and the other the so-called coulomb loss. This is akin to dry friction, where the energy loss depends only on the amplitude, not on the rate. A phenomenon such as this might be due to chain breakage or some similar mechanism. The natural rubber system appears quite complex. The elastic constant decreases with increasing frequency.

The work on attempting to analyze the actual equations of the analog is still in the early stages. The above conclusions concerning the nature of the rubber dynamic system is at best tentative and certainly subject to change. In our future work, we hope to explore this point more thoroughly.

APPENDIX I

The following analysis is in part that of Schechter et al.[3] We can rewrite eq. (3) in the form:

$$e(t) = f_1(0)S(t) + \int_0^t [df_1(\lambda)/d\lambda]S(t-\lambda)d\lambda \quad (4)$$

where the nomenclature is the same as that of eq. (3) of the text. For sine wave forcing eq. (4) becomes

$$e(t) = \int_0^t [df_1(\lambda)/d\lambda] \sin \omega(t-\lambda)d\lambda \quad (5)$$

since $f_1(0) = 0$.

Then eq. (5) can be rewritten

$$e(t) = \int_0^t [df_1(\lambda)/d\lambda] (\sin \omega t \cos \omega\lambda - \cos \omega t \sin \omega\lambda)d\lambda \quad (6)$$

On integration of eq. (6) by parts we get

$$e(t) = \omega \sin \omega t \int_0^t f_1(\lambda) \sin \omega\lambda d\lambda + \omega \cos \omega t \int_0^t f_1(\lambda) \cos \omega\lambda d\lambda \quad (7)$$

Now if we wait long enough so that we can reach a time t' such that at t' the response to the step has settled down to very close to its equilibrium value, eq. (7) can be written:

$$e(t) \cong \omega \sin \omega t \left[\int_0^{t'} f_1(\lambda) \sin \omega\lambda d\lambda + C\int_{t'}^t \sin \omega\lambda d\lambda\right] + \omega \cos \omega t \left[\int_0^{t'} f_1(\lambda) \cos \omega\lambda d\lambda + C\int_{t'}^t \cos \omega\lambda d\lambda\right] \quad (8)$$

Integrating and simplifying eq. (8) we obtain:

$$e(t) \cong \left[\omega \int_0^{t'} f_1(\lambda) \sin \omega\lambda d\lambda + C \cos \omega t'\right]\sin \omega t + \left[\omega \int_0^{t'} f_1(\lambda) \cos \omega\lambda d\lambda - C \sin \omega t'\right]\cos \omega t \quad (9)$$

However, we know that the response of the system after all transients have died down is of the form

$$e(t) = X \sin (\omega t - \Phi) \quad (10)$$

Comparing eqs. (9) and (10) we see that

$$X = (A^2 + B^2)^{1/2} \quad (11)$$

and

$$\Phi = \tan^{-1}(B/A)$$

where

$$A = \omega \int_0^{t'} f_1(\lambda) \sin \omega\lambda d\lambda + C \cos \omega t'$$
$$B = \omega \int_0^{t'} f_1(\lambda) \cos \omega\lambda d\lambda - C \sin \omega t' \quad (12)$$

Hence, if we can evaluate the expressions in eq. (12) for various frequencies, we will be in a position to calculate the response of the frequency range.

Schechter evaluates the integrals by various approximation formulae which must be computed on a digital computer. However, by use of a relatively small, table-top analog computer, it is possible to get A and B in a very direct fashion.

The following example illustrates the method quite directly. Figure 17 shows the response of a butyl sample as obtained from the step function machine. By means of the arbitrary function generator of the analog, this response was duplicated and is shown in Figure 18. Sine and

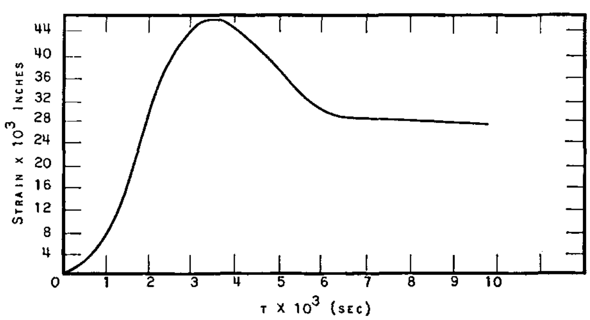

Fig. 17. Measured response to "step" of butyl.

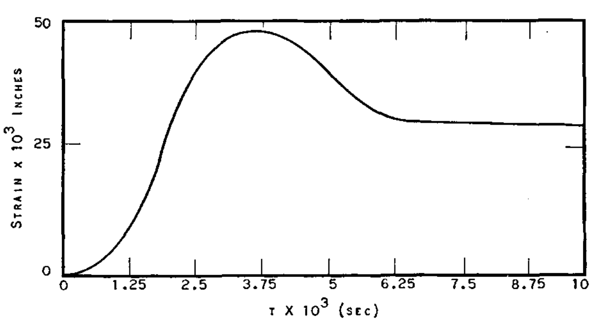

Fig. 18. Response as generated on function generator.

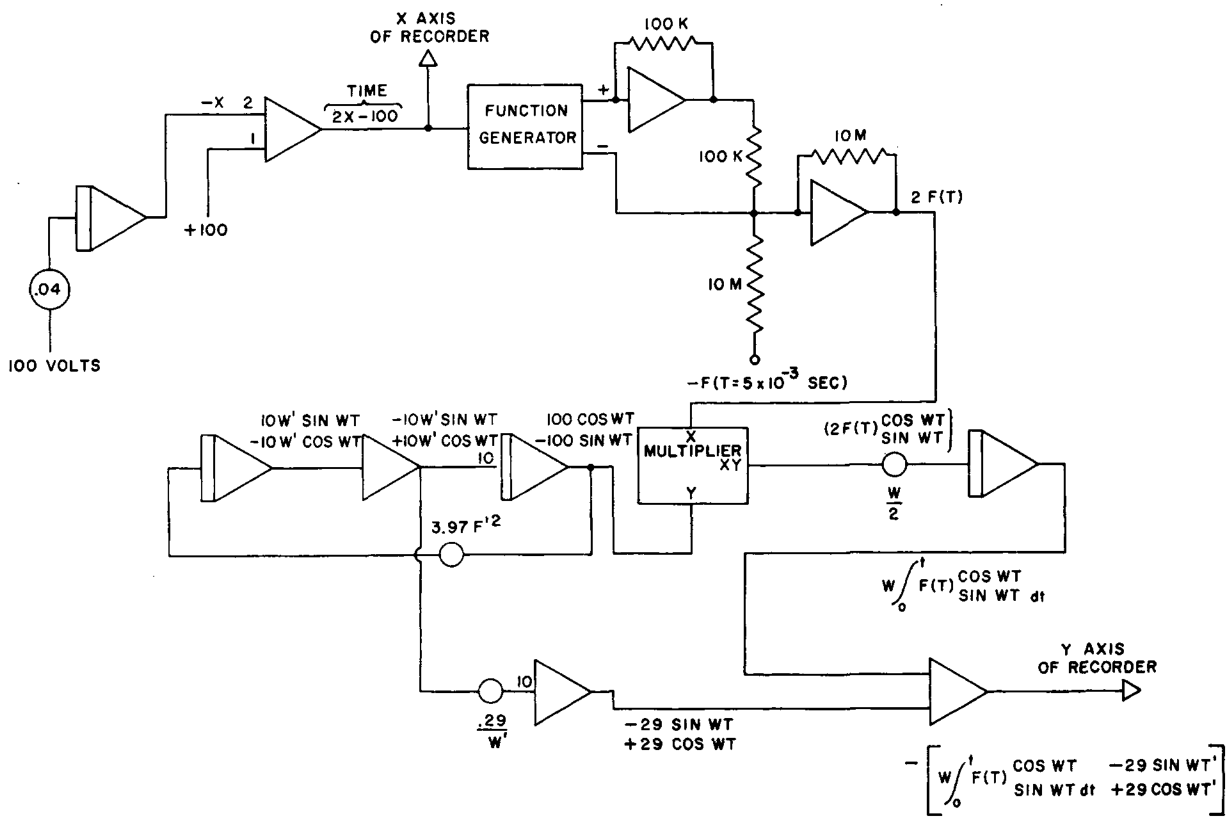

Fig. 19. Analog diagram for obtaining frequency response from step data.

cosine waves are generated on the analog and combined with the output of the function generator to form the expressions in eq. (12). These expressions are fed to an x–y recorder, where the second axis is driven by a time generator. The values of A and B are read at "real" time t' (in this case 10^{-2} sec.), and thus X and Φ can be calculated. The analog drawing is shown in Figure 19 and the result of the computation in Figure 20.

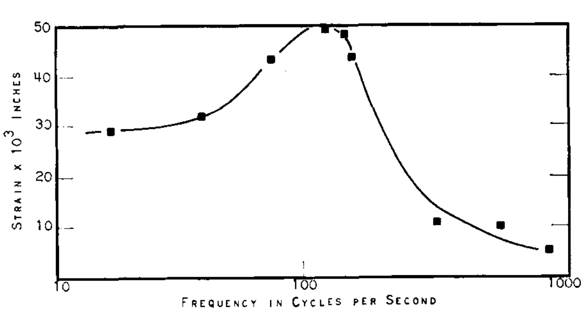

Fig. 20. Frequency response of butyl (calculated via technique of Appendix I).

APPENDIX II

In order to evaluate eq. (3) for nonsinusoidal driving forces, we have found it necessary to employ graphical integration. The following example illustrates the technique. Suppose we wish to find the response of the butyl discussed in Appendix I to a force described by Figure 21, i.e., a parabolic force. The equation describing this force is

$$S = (0.02t - t^2) \times 10^4 \tag{13}$$

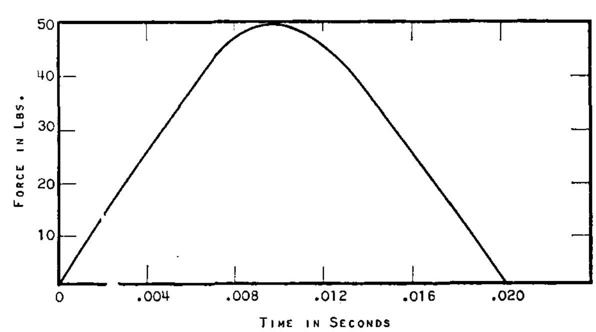

Fig. 21. Parabolic force.

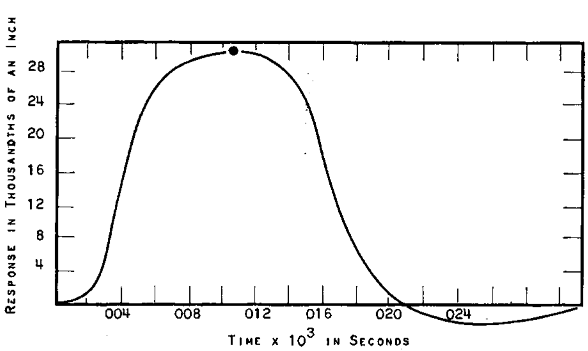

Fig. 22. Response for parabolic forcing.

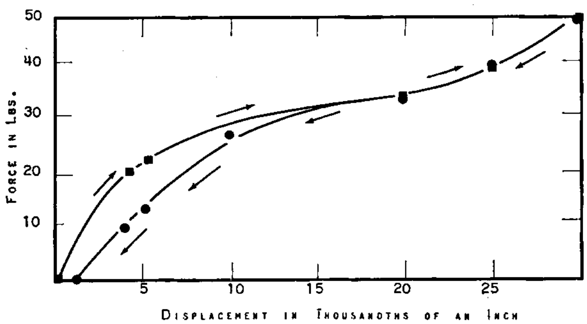

Fig. 23. Hysteresis loop for parabolic forcing.

(Actually this expression should be multiplied by a factor of 50. However, since the response to a "unit" is really the response to a step of 50 lb., unity in this case is equivalent to 50.) Inserting eq. (13) in eq. (3) of the text we have

$$e(t) = 10^4 \int_0^t f_1(t - \lambda)\ (0.02 - 2\lambda) d\lambda \quad (14)$$

where $f_1(t)$ is, of course, described by Figure 17. If we plot the product, $f_1(t_1 - \lambda)\ (0.02 - 2\lambda)$, against λ and then find the area under the curve, we have $e(t_1)$. Doing this for a large number of t_1 yields the response $e(t)$. This is shown in Figure 22. From Figures 21 and 22 it is possible to construct the stress–strain curve of the rubber under this type of forcing. This is shown in Figure 23. The area within the loop, of course, is the energy lost during deformation.

References

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Synopsis

The dynamic behavior of elastomers is usually studied in terms of a sinusoidal system. The theoretical and practical limitations of such analysis are discussed. A method which permits calculation of dynamic properties over a wide range of frequencies and forcing functions is described. The theoretical justification and limitations as well as the experimental setup are presented. Using SBR, natural rubber, and butyl, examples of the dynamic responses for various forcing functions are shown. The analogue computer is used to show properties of the above described rubbers which are useful from both the theoretical and application-oriented viewpoints.

Résumé

Le comportement au point de vue dynamique des élastomères a été étudié d'habitude suivant un système sinusoidal. Les limitations théoriques et pratiques d'une telle analyse sont discutées. Une méthode, qui permet le calcul des propriétés dynamiques dans un vaste domaine de fréquences et de fonctions forcées, est décrite. La justification et les limitations théoriques de même que le monde opératoire expérimental, sont présentées. En employant du caoutchouc SBR, du caoutchouc naturel et de butyle, des exemples de réponses dynamiques pour des fonctions forcées variées, sont donnés. Un calcul analogue est employé pour montrer les propriétés des caoutchoucs, décrits ci-dessus; ceci est intéressant pour l'usage tant du point de vue théorique que du point de vue des applications.

Zusammenfassung

Das dynamische Verhalten von Elastomeren wird gewöhnlich als das eines durch eine Sinusfunktion darstellbaren Systems untersucht. Die theoretischen und praktischen Beschränkungen einer solchen Analyse werden diskutiert. Eine Methode wird beschrieben, welche die Berechnung dynamischer Eigenschaften über einen weiten Bereich von Frequenzen und Beanspruchungsfunktionen gestattet. Die theoretische Begründung und die Begrenzungen der Methode, sowie die experimentelle Anordnung werden angegeben. Für SBR, Naturkautschuk, und Butyl werden Beispiele für das dynamische Verhalten bei verschiedenen Beanspruchungsfunktionen gegeben. Ein Analogcomputer wird zur Aufzeigung von Eigenschaften der oben beschriebenen Kautschuke verwendet, welche sowohl von einem theoretischen als auch anwendungstechnischen Gesichtspunkt aus interessant erscheinen.

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